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6.2 FRACTIONAL EXPONENTS AND RADICAL EXPRESSIONS

A *radical expression* is an expression involving roots. For example, $\sqrt{a}$ is the positive number whose square is a. Thus, $\sqrt{9} = 3$ since $3^2 = 9$, and $\sqrt{625} = 25$ since $25^2 = 625$. Similarly, the cube root of a, written $\sqrt[3]{a}$, is the number whose cube is a. So $\sqrt[3]{64} = 4$, since $4^3 = 64$. In general, the n^{th} root of a is the number b, such that $b^n = a$.

If $a < 0$, then the n^{th} root of a exists when n is odd, but is not a real number when n is even. For example, $\sqrt[3]{-27} = -3$, because $(-3)^3 = -27$, but $\sqrt{-9}$ is not defined, because there is no real number whose square is -9.

Example 1

Evaluate:

(a) $\sqrt{121}$

(b) $\sqrt[5]{-32}$

(c) $\sqrt{-4}$

(d) $\sqrt[4]{\dfrac{81}{625}}$

Solution

(a) Since $11^2 = 121$, we have $\sqrt{121} = 11$.

(b) The fifth root of -32 is the number whose fifth power is -32. Since $(-2)^5 = -32$, we have $\sqrt[5]{-32} = -2$.

(c) Since the square of a real number cannot be negative, $\sqrt{-4}$ does not exist.

(d) Since

$$\left(\frac{3}{5}\right)^4 = \frac{81}{625}, \quad \text{we have} \quad \sqrt[4]{\frac{81}{625}} = \frac{3}{5}.$$

Using Fractional Exponents to Describe Roots

The laws of exponents suggest an exponential notation for roots involving fractional exponents. For instance, applying the exponent rules to the expression $a^{1/2}$, we get

$$(a^{1/2})^2 = a^{(1/2)\cdot 2} = a^1 = a.$$

Thus, $a^{1/2}$ should be the number whose square is a, so we define

$$a^{1/2} = \sqrt{a}.$$

Similarly, we define

$$a^{1/3} = \sqrt[3]{a} \quad \text{and} \quad a^{1/n} = \sqrt[n]{a}.$$

The Exponent Laws Work for Fractional Exponents

The exponent laws also work for fractional exponents.

Example 2

Evaluate

(a) $25^{1/2}$

(b) $9^{-1/2}$

(c) $8^{1/3}$

(d) $27^{-1/3}$.

Solution

(a) We have $25^{1/2} = \sqrt{25} = 5$.

(b) Using the rules about negative exponents, we have

$$9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$$

(c) Since $2^3 = 8$, we have $8^{1/3} = \sqrt[3]{8} = 2$.

(d) Since $3^3 = 27$, we have

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}.$$

Fractional Exponents with Numerators Other Than One

We can also use the exponent rule $(a^n)^m = a^{nm}$ to define the meaning of fractional exponents in which the numerator of the exponent is not 1.

Example 3

Find

(a) $64^{2/3}$

(b) $9^{-3/2}$

(c) $(-216)^{2/3}$

(d) $(-625)^{3/4}$

Solution

(a) Writing $2/3 = 2 \cdot (1/3)$ we have

$$64^{2/3} = 64^{2 \cdot (1/3)} = \left(64^2\right)^{1/3} = \sqrt[3]{64^2} = \sqrt[3]{4096} = 16.$$

We could also do this the other way around, and write $2/3 = (1/3) \cdot 2$:

$$64^{2/3} = 64^{(1/3) \cdot 2} = \left(64^{1/3}\right)^2 = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16.$$

(b) We have

$$9^{-3/2} = \frac{1}{9^{3/2}} = \frac{1}{(\sqrt{9})^3} = \frac{1}{3^3} = \frac{1}{27}.$$

(c) To find $(-216)^{2/3}$, we can first evaluate $(-216)^{1/3} = -6$, and then square the result. This gives $(-216)^{2/3} = (-6)^2 = 36$.

(d) Writing $(-625)^{3/4} = ((-625)^{1/4})^3$, we conclude that $(-625)^{3/4}$ is not a real number since $(-625)^{1/4}$ is an even root of a negative number.

Example 4

Write each of the following as an equivalent expression in the form x^n and give the value for n.

(a) $\dfrac{1}{x^3}$

(b) $\sqrt[5]{x}$

(c) $(\sqrt[3]{x})^2$

(d) $\sqrt{x^5}$

(e) $\dfrac{1}{\sqrt[4]{x}}$

(f) $\left(\dfrac{1}{\sqrt{x}}\right)^3$

Solution

(a) We have

$$\frac{1}{x^3} = x^{-3} \quad \text{so} \quad n = -3.$$

(b) We have $\sqrt[5]{x} = x^{1/5}$, so $n = 1/5$.

(c) We have $(\sqrt[3]{x})^2 = (x^{1/3})^2$. According to the exponent laws we multiply the exponents in this expression, so $(x^{1/3})^2 = x^{2/3}$, so $n = 2/3$.

(d) We have $\sqrt{x^5} = (x^5)^{1/2} = x^{5/2}$, so $n = 5/2$.

(e) We have

$$\frac{1}{\sqrt[4]{x}} = \frac{1}{x^{1/4}} = x^{-1/4} \quad \text{so} \quad n = -1/4.$$

(f) We have

$$\left(\frac{1}{\sqrt{x}}\right)^3 = \left(\frac{1}{x^{1/2}}\right)^3 = (x^{-1/2})^3 = x^{-3/2} \quad \text{so} \quad n = -3/2.$$

Example 5

Simplify each expression, assuming all variables are positive.

(a) $\sqrt[3]{2}\sqrt[3]{4}$

(b) $\sqrt[3]{z^6w^9}$

Solution

(a) Using the fact that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, we have $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$.

(b) We can write $\sqrt[3]{z^6w^9}$ as $(z^6w^9)^{1/3} = z^2w^3$.

Working with Radical Expressions

Since roots are really fractional powers, our experience with integer powers leads us to expect that expressions involving roots of sums and differences will not be as easy to simplify as expressions involving roots of products and quotients.

Example 6

Do the expressions have the same value?

(a) $\sqrt{9+16}$ and $\sqrt{9}+\sqrt{16}$

(b) $\sqrt{100-64}$ and $\sqrt{100}-\sqrt{64}$

(c) $\sqrt{(9)\,(4)}$ and $\sqrt{9}\cdot\sqrt{4}$

(d) $\sqrt{\dfrac{100}{4}}$ and $\dfrac{\sqrt{100}}{\sqrt{4}}$

Solution

(a) We have $\sqrt{9+16}=\sqrt{25}=5$, but $\sqrt{9}+\sqrt{16}=3+4=7$, so the expressions have different values. In general, $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$.

(b) Here, $\sqrt{100-64}=\sqrt{36}=6$. However, $\sqrt{100}-\sqrt{64}=10-8=2$. In general, $\sqrt{a-b}\neq\sqrt{a}-\sqrt{b}$.

(c) We have $\sqrt{(9)(4)}=\sqrt{36}=6$. Also, $\sqrt{9}\cdot\sqrt{4}=3\cdot 2=6$. In general, the square root of a product is equal to the product of the square roots, or $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$.

(d) We have $\sqrt{\dfrac{100}{4}}=\sqrt{25}=5$. Also, $\dfrac{\sqrt{100}}{\sqrt{4}}=\dfrac{10}{2}=5$. In general, $\sqrt{\dfrac{a}{b}}=\dfrac{\sqrt{a}}{\sqrt{b}}$.

Simplifying Radical Expressions That Contain Sums and Differences

Although the laws of exponents do not tell us how to simplify expressions involving roots of sums and differences, there are other methods that work. Again, we start with a numerical example.

Example 7

Are the expressions equivalent?

(a) $\sqrt{100}+\sqrt{100}$ and $\sqrt{200}$

(b) $\sqrt{100}+\sqrt{100}$ and $2\sqrt{100}$

Solution

(a) We know that $\sqrt{100}+\sqrt{100}=10+10=20$. But, the square root of 200 cannot be 20, since $20^2 = 400$, so they are not equivalent.

(b) They are equivalent, since $\sqrt{100}+\sqrt{100}=10+10=20$ and $2\sqrt{100}-2(10)=20$. Another way to see this without evaluating is to use the distributive law to factor out the common term $\sqrt{100}$:

$$\sqrt{100}+\sqrt{100}=\sqrt{100}(1+1)=2\sqrt{100}.$$

Expressing roots as fractional powers helps determine the principles for combining like terms in expressions involving roots. The principles are the same as for integer powers: we can combine terms involving the same base and the same exponent.

Example 8

Combine the radicals if possible.

(a) $3\sqrt{7}+6\sqrt{7}$

(b) $4\sqrt{x}+3\sqrt{x}-\sqrt{x}$

(c) $9\sqrt[3]{2}-5\sqrt[3]{2}+\sqrt{2}$

(d) $-5\sqrt{5}+8\sqrt{20}$

(e) $5\sqrt{4x^3}+3x\sqrt{36x}$

Solution

(a) We have $3\sqrt{7}+6\sqrt{7}=9\sqrt{7}$.

(b) We have $4\sqrt{x}+3\sqrt{x}-\sqrt{x}=6\sqrt{x}$.

(c) Expressing the roots in exponential notation, we see that not all the exponents are the same:

$$9\sqrt[3]{2}-5\sqrt[3]{2}+\sqrt{2}=9\cdot 2^{1/3}-5\cdot 2^{1/3}+2^{1/2}=4\cdot 2^{1/3}+2^{1/2}=4\sqrt[3]{2}+\sqrt{2}.$$

(d) Here there is a term involving $\sqrt{5}=5^{1/2}$ and a term involving $\sqrt{20}=20^{1/2}$. Thus the exponents are the same in both terms, but the bases are different. However, $8\sqrt{20}=8\sqrt{(4)(5)}=8\sqrt{4}\sqrt{5}=8\cdot 2\sqrt{5}=16\sqrt{5}$. Therefore,

$$-5\sqrt{5}+8\sqrt{20}=-5\sqrt{5}+16\sqrt{5}=11\sqrt{5}.$$

(e) Expressing each term in exponential form, we get

$$5\sqrt{4x^3}+3x\sqrt{36x}=5(4x^3)^{1/2}+3x(36x)^{1/2}=5\cdot 4^{1/2}(x^3)^{1/2}+3x(36)^{1/2}x^{1/2}$$
$$=10x^{3/2}+18x^{3/2}=28x^{3/2}.$$

Rationalizing the Denominator

In Section 2.3 we saw how to simplify fractions by taking a common factor out of the numerator and denominator. Sometimes it works the other way around: we can simplify a fraction with a radical expression in the denominator, such as

$$\frac{\sqrt{2}}{2+\sqrt{2}},$$

by multiplying the numerator and denominator by a carefully chosen common factor.

Example 9 Simplify $\frac{\sqrt{2}}{2+\sqrt{2}}$ by multiplying the numerator and denominator by $2-\sqrt{2}$.

The process in Example 9 is called *rationalizing the denominator*. In general, to rationalize a sum of two terms, one or more of which is a radical, we multiply it by the sum obtained by changing the sign of one of the radicals. The resulting sum is a *conjugate* of the original sum.

Example 10

Multiply each expression by a conjugate.

(a) $1+\sqrt{2}$

(b) $-6-2\sqrt{7}$

(c) $3\sqrt{5}-2$

(d) $\sqrt{3}+3\sqrt{6}$

Solution

(a) The conjugate of $1+\sqrt{2}$ is $1-\sqrt{2}$, and the product is

$$(1+\sqrt{2})\,(1-\sqrt{2})=1^2-(\sqrt{2})^2=1-2=-1.$$

(b) The conjugate of $-6-2\sqrt{7}$ is $-6+2\sqrt{7}$, and the product is

$$(-6-2\sqrt{7})\,(-6+2\sqrt{7})=(-6)^2-(2\sqrt{7})^2=36-28=8.$$

(c) The conjugate of $3\sqrt{5}-2$ is $-3\sqrt{5}-2$. We rewrite the product so it is easier to see the difference of two squares:

$$(3\sqrt{5}-2)\,(-3\sqrt{5}-2)=(-2+3\sqrt{5})\,(-2-3\sqrt{5})=(-2)^2-(3\sqrt{5})^2=4-45=-41.$$

(d) Here there are two radicals, so there are two possible conjugates, $\sqrt{3}-3\sqrt{6}$ and $-\sqrt{3}+3\sqrt{6}$. We choose the first one:

$$(\sqrt{3}+3\sqrt{6})(\sqrt{3}-3\sqrt{6})=(\sqrt{3})^2-(3\sqrt{6})^2=3-54=-51.$$

Problems for Section 6.2

EXERCISES

Evaluate the expressions in Exercises 1, 2, 3 and 4 without using a calculator.

1. $4^{1/2}$

Answer:

2

2. $25^{-1/2}$

3. $\left(\frac{4}{9}\right)^{-1/2}$

Answer:

3/2

4. $\left(\frac{64}{27}\right)^{-1/3}$

In Exercises 5 and 6, write each expression without parentheses. Assume all variables are positive.

5. $(2^{4x}5^{4x})^{1/2}$

Answer:

100^x

6. $\left(\frac{10^{6a}}{5^{6a}}\right)^{1/3}$

Simplify the expressions in Exercises 7, 8, 9 and 10, assuming all variables are positive.

7. $\sqrt[5]{\frac{x^{10}}{y^5}}$

Answer:

$\frac{x^2}{y}$

8. $\sqrt[4]{4x^3}\sqrt[4]{4x^5}$

9. $\sqrt{48a^3b^7}$

Answer:

$4ab^2\sqrt{3ab}$

10. $\frac{\sqrt[3]{96x^7y^8}}{\sqrt[3]{3x^4y}}$

In Exercises 11, 12, 13, 14, 15 and 16, write the expression as an equivalent expression in the form x^n and give the value for n.

11. $\frac{1}{\sqrt{x}}$

Answer:

$x^{-1/2}$; $n = -1/2$

12. $\frac{1}{x^5}$

13. $\sqrt{x^3}$

Answer:

$x^{3/2}$; $n = 3/2$

14. $(\sqrt[3]{x})^5$

15. $1/(1/x^{-2})$

Answer:

x^{-2}, $n = -2$

16. $\dfrac{(x^3)^2}{(x^2)^3}$

In Exercises 17, 18, 19, 20, 21, 22, 23, 24, 25 and 26, combine radicals, if possible.

17. $12\sqrt{11} - 3\sqrt{11} + \sqrt{11}$

Answer:

$10\sqrt{11}$

18. $5\sqrt{9} - 2\sqrt{144}$

19. $2\sqrt{3} + \dfrac{\sqrt{3}}{2}$

Answer:

$5\sqrt{3}/2$

20. $\sqrt[3]{4x} + 6\sqrt[3]{4x} - 2\sqrt[3]{4x}$

21. $8\sqrt[3]{3} - 2\sqrt[3]{3} - 2\sqrt{3}$

Answer:

$2\left(3\sqrt[3]{3} - \sqrt{3}\right)$

22. $-6\sqrt{98} + 4\sqrt{8}$

23. $6\sqrt{48a^4} + 2a\sqrt{27a^2} - 3a^2\sqrt{75}$

Answer:

$15a^2\sqrt{3}$

24. $5\sqrt{12t^3} + 2t\sqrt{128t} - 3t\sqrt{48t}$

25. $\dfrac{\sqrt{45}}{5} - \dfrac{2\sqrt{20}}{5} + \dfrac{\sqrt{80}}{\sqrt{25}}$

Answer:

$\dfrac{3\sqrt{5}}{5}$

26. $4xy\sqrt{90xy} + 2\sqrt{40x^3y^3} - 3xy\sqrt{50xy}$

In Exercises 27, 28, 29, 30, 31 and 32, find a conjugate of each expression and the product of the expression with the conjugate.

27. $4 + \sqrt{6}$

Answer:

10

28. $\sqrt{13} - 10$

29. $-\sqrt{5} - \sqrt{6}$

Answer:

-1

30. $7\sqrt{2} - 2\sqrt{7}$

31. $b\sqrt{a}+a\sqrt{b}$

Answer:

ab^2 - a^2b

32. $1-\sqrt{r+1}$

In Exercises 33, 34, 35, 36 and 37, rewrite each expression by rationalizing the denominator.

33. $\dfrac{2}{\sqrt{3}+1}$

Answer:

$\sqrt{3}-1$

34. $\dfrac{\sqrt{5}}{5-\sqrt{5}}$

35. $\dfrac{10}{\sqrt{6}-1}$

Answer:

$2(\sqrt{6}+1)$

36. $\dfrac{\sqrt{3}}{3\sqrt{2}+\sqrt{3}}$

37. $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$

Answer:

$2+\sqrt{3}$

PROBLEMS

38. The surface area (not including the base) of a right circular cone of radius r and height $h > 0$ is given by

$$\pi r\sqrt{r^2+h^2}.$$

Explain why the surface area is always greater than πr^2

(a) In terms of the structure of the expression.

(b) In terms of geometry.

By giving specific values for a, b, and c, explain how the exponent rule

$$(a^b)^c=a^{bc}$$

is used to rewrite the expressions in Problems 39 and 40.

39. $(2m^2n^4)^{3r+3}=(8m^6n^{12})^{r+1}$

40. $\sqrt{(x+1)\ (x^2+2x+1)}=(x+1)^{3/2}$