Unit 3 Review

Key Ideas

- Definition of Derivative
- Notation
- Relationship between Graphs of \( f \) and \( f' \)
- Graphing the derivative from data
- One sided derivatives
- Differentiability implies local linearity
- Derivatives on a calculator
- Differentiability implies continuity
- Intermediate Value Theorem for Derivatives
- Differentiability rules
- Instantaneous rates of change
- Motion along a line
- Slopes of Parametrized Curves
- Implicit functions
- Derivatives of trigonometric functions
- Derivatives of exponential and logarithmic functions
- Derivatives of inverse functions

Concepts

**Intermediate Value Theorem for Derivatives**

If \( a \) and \( b \) are any two points in an interval on which \( f \) is differentiable, then \( f'' \) takes on every value between \( f'(a) \) and \( f'(b) \).

\[
\frac{d}{dx}(c) = 0 \\
\frac{d}{dx}(x^n) = nx^{n-1} \\
\frac{d}{dx}(cu) = c \frac{du}{dx} \\
\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \\
\frac{d}{dx}(uv) = \frac{du}{dx}v + u \frac{dv}{dx} \\
\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{du}{dx} \frac{v - u \frac{dv}{dx}}{v^2} \\
\]

\[
\frac{dv}{dx} = \frac{dy}{ux} \\
\frac{du}{dx} = \frac{dy}{dx} \\
\]
**Inverse function**

\[ g'(x)|_{x=c} = \frac{1}{f'(y)|_{y=a}} \quad \text{where } y = f(x) \text{ and } g(x) = f^{-1}(x) \text{ and } f(a) = c \text{ and } g(c) = a. \]

\[ \frac{d}{dx} (\cos(x)) = -\sin(x) \]
\[ \frac{d}{dx} (\sin(x)) = \cos(x) \]
\[ \frac{d}{dx} (\tan(x)) = \sec^2(x) \]
\[ \frac{d}{dx} (\cot(x)) = -\csc^2(x) \]
\[ \frac{d}{dx} (\sec(x)) = \sec(x)\tan(x) \]
\[ \frac{d}{dx} (\csc(x)) = -\csc(x)\cot(x) \]

\[ \frac{d}{dx} (\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1 \]
\[ \frac{d}{dx} (\cos^{-1}(u)) = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad |u| < 1 \]
\[ \frac{d}{dx} (\tan^{-1}(u)) = \frac{1}{1+u^2} \frac{du}{dx} \]

\[ \frac{d}{dx} (e^u) = e^u \frac{du}{dx} \]
\[ \frac{d}{dx} (a^u) = a^u \ln(a) \frac{du}{dx} \]
\[ \frac{d}{dx} (\ln(u)) = \frac{1}{u} \frac{du}{dx} \]
\[ \frac{d}{dx} (\log_a u) = \frac{1}{u \ln(a)} \frac{du}{dx} \]
AP Specific Practice Question

1. If \( f(x) = 4x^2 + 5 \), then \( f(9) = \)

2. If \( f(x) = \frac{4x^2 + x}{4x^2 - x} \), then \( f''(x) = \)

3. If \( x^2 - 3xy + 2y^2 = 16 \), then \( \frac{dy}{dx} = \)

4. If \( f(x) = \csc(x) - \sec(x) \), then \( f'(x) = \)

5. An equation of the line normal to the curve of the graph \( y = \sqrt{2x^2 + 7x} \) at \( (1, 3) \) is

6. If \( f(x) = 3^{5x} \), then \( f''(x) = \)

7. If \( f(x) = 4x^2 - x \) and \( g(x) = f^{-1}(x) \), then \( g'(5) \) could be

8. If the function \( f(x) \) is differentiable and \( f(x) = \begin{cases} ax^3 - 5x, & x \leq 1 \\ bx^2 + 3, & x > 1 \end{cases} \) then \( a = \)

9. Two objects leave the origin at the same time and move along the y-axis with their respective positions determined by the functions \( y_1 = 4 \cos(t) \) and \( y_2 = \sin(2t) \) for \( 0 < t < 5 \). For how many values of \( t \) do the particles have the same acceleration? [you will need a calculator for this question].

10. The curve below is the graph of \( y = k(s) \) on the domain \([0, 4]\).

Which of the following statements about \( k \) are true?

i) There is some point \( c \) in \([0, 4]\) such that \( k'(c) = 0 \)

ii) \( k \) is continuous at \( s=3 \)

iii) \( k'(\frac{1}{2}) > 0 \)
Free Response Questions
(Obtain questions from website, some questions you will not be able to complete all section)

Question 4 1998
Question 6 1998
Question 6 1999
Question 5 2000
Question 2 2003
Unit 3.15 Solutions

1. \[ f(x) = 4x^{\frac{5}{2}} \]
   \[ f'(x) = \frac{5}{2}x^{\frac{3}{2}} \]
   \[ f''(9) = \frac{5}{2}(27) = 67.5 \]

2. \[ f(x) = \frac{4x^2 + x}{4x^2 - x} \]
   \[ f'(x) = \frac{[8x+1](4x^2-x)-(4x^2+x)[8x-1]}{(4x^2-x)^2} \]
   \[ = \frac{32x^3 - 4x^2 - x - (32x^3 + 4x^2 - 1)}{(4x^2-x)^2} \]
   \[ = \frac{-8x^2}{(4x^2-x)^2} \]
   \[ = -\frac{8}{(4x-1)} \]

3. \[ x^2 - 3xy + 2y^2 = 16 \]
   \[ 2x - \left[ 3(y) + 3x \left( \frac{dy}{dx} \right) + 4y \frac{dy}{dx} \right] = 0 \]
   \[ 2x - 3y - 3x \frac{dy}{dx} + 4y \frac{dy}{dx} = 0 \]
   \[ -3x \frac{dy}{dx} + 4y \frac{dy}{dx} = 3y - 2x \]
   \[ \frac{dy}{dx} \left[ -3x + 4y \right] = 3y - 2x \]
   \[ \frac{dy}{dx} = \frac{3y - 2x}{-3x + 4y} \]

4. \[ f(x) = \csc(x) - \sec(x) \]
   \[ f'(x) = -\csc(x) \cot(x) - \sec(x) \tan(x) \]
Let’s find the slope of the tangent line.

\[ y = \sqrt{2x^2 + 7x} \]
\[ y' = \frac{4x + 7}{2\sqrt{2x^2 + 7x}} \]
\[ y'|_{x=1} = \frac{11}{6} \]

The slope of the normal line is \( m = -\frac{6}{11} \).

The equation of the line is

\[ y - 3 = -\frac{6}{11} (x-1) \]
\[ y = -\frac{6}{11} x + \frac{39}{11} \]

\[ f(x) = 3^{5x} \]
\[ f'(x) = 3^{5x} \cdot \ln(3) \cdot 5 \]

Recall the derivative of an inverse function is \( g'(x) = \frac{1}{f''(y)} \) where \( y = f(x) \) and \( g(x) = f^{-1}(x) \).

So we first need the value of \( x \) that corresponds to \( y=5 \)

\[ 5 = 4x^2 - x \]
\[ 4x^2 - x - 5 = 0 \]
\[ (x+1)(4x-5) = 0 \]
\[ \therefore x = -1, \ x = \frac{5}{4} \]

Plugging into \( \frac{1}{f'(y)} = \frac{1}{8y-1} \)

\[ \frac{1}{8(-1)-1} = -\frac{1}{9} \quad \text{or} \quad \frac{1}{8\left(\frac{5}{4}\right)-1} = \frac{1}{9} \]

either answer is valid.
8. Since differentiable, the derivative of each piecewise function must equal each other, specifically at \( x=1 \). And since differentiable, the piecewise function is also continuous, specifically at \( x=1 \).

\[
 f(x) = \begin{cases} 
 ax^3 - 5x, & x \leq 1 \\
 bx^2 + 3, & x > 1 
\end{cases}
\]

\[
 f'(x) = \begin{cases} 
 3ax^2 - 5, & x \leq 1 \\
 2bx, & x > 1 
\end{cases}
\]

From \( f(1) \): \( a - 5 = b + 3 \)
From \( f'(1) \): \( 3a - 5 = 2b \)

Solving: \( 3a - 5 = 2(a - 8) \)
\[ 3a - 5 = 2a - 16 \]
\[ a = -11 \]

9. \[
 \frac{dy_1}{dt} = -4\sin(t) \quad \frac{dy_2}{dt} = 2\cos(2t) 
\]
\[
 \frac{d^2 y_1}{dt^2} = -4\cos(t) \quad \frac{d^2 y_2}{dt^2} = -4\sin(2t) 
\]

Now all we have to do is graph both of these second derivatives within the domain and count the number of intersection points.

The answer is 4.

10. Since the graph is never horizontal, then only ii) and iii) are correct.